1.3a First order linear difference equations

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Consider a first-order linear difference eqn

$$x_{t+1} = a_t x_t + b_t$$

Given initial condition x_0, this has a unique solution

$$x_{t+1} = \left(\prod_{i=0}^{t} a_i\right) x_0 + b_t + \sum_{i=0}^{t-1} b_i \prod_{j=i+1}^{t} a_j$$

Suppose $a_t \equiv a$ and $b_t \equiv b$ constants. Then

$$x_{t+1} = a x_t + b$$

$$x_{t+1} = a^{t+1} x_0 + b \sum_{i=0}^{t} a^i$$

$$\Rightarrow \quad x_{t+1} = \begin{cases} x_0 a^{t+1} + b \cdot \dfrac{1 - a^{t+1}}{1 - a}, & a \neq 1 \\ x_0 + (t+1) b, & a = 1. \end{cases}$$

Principle of Superposition

Suppose $x_{h,t}$ satisfies $x_{h,t+1} = a x_{h,t}$

Suppose $x_{p,t}$ satisfies $x_{p,t+1} = a x_{p,t} + b$

Then $x_t = x_{h,t} + x_{p,t}$ satisfies $x_{t+1} = ax_t + b$ by linearity.

Method of undetermined coefficients

Note $x_{h,t} = ca^t$ solves $x_{h,t+1} = ax_{h,t}$ $\forall\, c \in \mathbb{R}$.

Need to find x_p. Guess $x_p = k$ for some constant k.

If $a \neq 1$, this works, and $k = ak + b \Rightarrow k = \frac{b}{1-a}$

If $a = 1$, guess $x_p = tk$. Then $(t+1)k = tk + b$

$\Rightarrow k = b$

So $x_t = \begin{cases} ca^t + \frac{b}{1-a}, & a \neq 1 \\ c + tb, & a = 1 \end{cases}$ solves $x_{t+1} = ax_t + b$

(general solution)

Cobwebbing

Suppose $x_{t+1} = f(x_t)$, a first-order equation

Then $y = f(x)$ is the reproduction curve.

Perform the following algorithm:

- Graph the lines $y = x$ and $y = f(x)$
- Let $(a, b) = (x_0, 0)$ be the starting point
- Repeat the following steps while moving to the new (a, b)
  - $b \leftarrow f(a)$

- $a \leftarrow b$